The uncertainty principle does *not* entirely determine the non-locality of quantum theory

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One of the most intriguing discoveries regarding quantum non-local correlations in recent years was the establishment of a direct correspondence between the quantum value of non-local games and the strength of the fine-grained uncertainty relations in *Science, vol. 330, no. 6007, 1072 (2010)*. It was shown that while the degree of non-locality in any theory is generally determined by a combination of two factors - the strength of the uncertainty principle and the degree of steering allowed in the theory, the most paradigmatic games in quantum theory have degree of non-locality purely determined by the uncertainty principle alone. In this context, the fundamental question arises: is this a universal property of optimal quantum strategies for all non-local games? Indeed, the above mentioned feature occurs in surprising situations, even when the optimal strategy for the game involves non-maximally entangled states. However, here we definitively prove that the answer to the question is negative, by presenting explicit counter-examples of non-local games and fully analytical optimal quantum strategies for these, where a definite trade-off between steering and uncertainty is absolutely necessary. We provide an intuitive explanation in terms of the Hughston-Jozsa-Wootters theorem [13] for when the relationship between the uncertainty principle and the quantum game value breaks down.

Introduction. The results of measurements on distant quantum systems can be correlated in a way that defies a classical local realistic description. This non-locality of quantum theory is evidenced in the violation of Bell inequalities by spatially separated quantum systems. The quantum correlations are restricted to some extent by the no-signaling principle, i.e., the measurement results cannot allow for signaling between the distant locations. Nevertheless, since [2] it is known that there are non-local correlations (now known as Popescu-Rohrlich boxes) allowed by the no-signaling principle that cannot be realized in quantum theory (see [3] for a generalization of this statement to all extremal no-signaling boxes). The question why quantum correlations are non-local yet not as strong as allowed by the no-signaling principle is an intriguing one and has stimulated the formulation of many striking information-theoretic principles of Nature. Yet so far none of the known principles has been able to capture the set of quantum correlations in its entirety [4], so that a comprehensive answer to this question is still lacking. On the other hand, although deviating slightly from the purely information-theoretic paradigm, a significant breakthrough was achieved in [1] where two fundamental concepts of quantum theory, namely the strength of non-local correlations and the uncertainty principle, were shown to be inextricably quantitatively linked with each other.

The uncertainty principle is a fundamental feature of quantum theory, which postulates the existence of incompatible observables such as position and momentum or the different spin components of a spin-1/2 particle, the results of whose measurements on identically prepared systems cannot be predicted simultane-

ously with certainty. Recently, the traditional formulation of uncertainty relations in term of standard deviations and commutators has been eschewed in favor of the entropic uncertainty relations [5] and going beyond these to the yet more fundamental fine-grained uncertainty relations. The fine-grained uncertainty relations are formulated in terms of the basic entities of the theory, namely the probabilities of particular sets of outcomes for given sets of measurements (different measurements being performed on identically prepared systems), and are thus able to capture the uncertainty of these measurements in a more general manner than the entropic measures or the statistical standard deviations. Moreover, the bounds on the probabilities are expressed in a manner independent of the specific underlying quantum state, typically being obtained by a maximization over all states, an advantage over the formulation of uncertainty relations in terms of average values of commutators for fixed states.

One concept which plays a crucial role in linking uncertainty relations and quantum non-locality is steering, identified by Schrödinger in [11]. This property determines which states Alice can prepare on Bob's system by a measurement on her part of a shared system. Suppose Alice and Bob share a composite system in a state $\hat{\sigma}_{AB}$, and Alice performs a measurement labeled x and obtains outcome a on her part of the system, then in theories where the state space is convex, Bob's state $\hat{\sigma}_B$ can be written as

$$\hat{\sigma}_B = \sum_a p(a|x)\hat{\sigma}_{a|x}^B, \qquad (1)$$

where $\hat{\sigma}_{a|x}^B$ belongs to Bob's state space; the combina-

tion $\{p(a|x), \hat{\sigma}_{a|x}^{B}\}$ is referred to as an "assemblage" [6]. Note that this does not violate the no-signaling principle, since for any choice of Alice's measurement x, Bob's state $\hat{\sigma}_{B}$ obtained by averaging over Alice's measurement outcomes a remains the same.

The intriguing result of [1] is that while the degree of non-locality in any theory is generally determined by a combination of two factors - the strength of the uncertainty principle and the degree of steering allowed in the theory, *in quantum theory the degree of non-locality is purely determined by the strength of the uncertainty principle alone.* More precisely, [1] shows that in a two-party Bell scenario, the strength of non-locality in any theory is determined by the uncertainty relations for Bob's measurements acting on the states that Alice can steer to. On the other hand in quantum theory, *for all games where the optimal measurements were known*, it is shown that the states which Alice can steer to are identical to the most certain states, so that only the uncertainty relations of Bob's measurements determine the outcome. This interesting discovery between the two fundamental properties of quantum theory begs for further investigation to identify if it holds for all two-party Bell inequalities, and whether it can be extended to include multi-party Bell inequalities as well. The tremendous difficulty in finding optimal quantum strategies for general non-local games has so far prevented progress in this direction.

In this paper, we present explicit counterexamples of non-local games and fully analytical optimal states and measurements for the same that show that the relation between the uncertainty principle and the degree of non-locality in quantum theory *does not* hold in general. The non-local games we present involve two parties with binary inputs and binary outputs but differ from the well-known CHSH game in one crucial respect - for at least one pair of inputs, the winning constraint stipulates that Alice and Bob return a single unique pair of outputs. Thus, the games do not belong to the class of XOR games, and can be thought of as hybrid versions of the CHSH game and the Hardy paradox. In particular, the games share the property with the Hardy paradox [7] that their optimal value is only achieved by a *non-maximally entangled* state. We prove a self-testing property of the above games, i.e., that essentially there is a unique optimal state and measurements (up to local unitaries and attaching irrelevant ancillas) that achieves the maximal quantum value of the game. While perfect steering is usually associated with the maximally entangled state, one could expect a violation of the relation in the cases of games with optima achieved by non-maximally entangled states. We therefore investigate this question by considering the case of the CGLMP Bell inequality and find that in spite of the fact that its maximal violation is only achieved with a non-maximally entangled state [14], Alice is able to steer Bob's state to the

maximally certain states for the fine-grained uncertainty relations in question. We provide an intuitive explanation in terms of the Hughston-Jozsa-Wootters theorem [13] for when the relationship between the uncertainty principle and the quantum game value (UP-QGV correspondence) breaks down.

Framework for the link between the quantum value of a game and uncertainty relations. In this section, we elaborate upon the statements made in the Introduction and recall the exact relationship between the fine-grained uncertainty relations and the strength of non-locality established in [1]. Consider a general two-party non-local game g, in which Alice and Bob receive questions x, y from respective input sets $\mathcal{X}, \mathcal{Y}$ according to some input distribution $\pi_{AB}(x, y)$. They return answers a, b from some output sets $\mathcal{A}, \mathcal{B}$, respectively, with the objective to maximize the success probability in the game $\omega(g)$. The winning constraint on the outputs for each pair of inputs is specified by a predicate $V(a, b|x, y)$; usually $V(a, b|x, y) \in \{0, 1\}$. The general Bell expression associated to a non-local game is thus written as

$$\sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \pi_{AB}(x, y) \sum_{\substack{a \in \mathcal{A} \\ b \in \mathcal{B}}} V(a, b|x, y) P(a, b|x, y) \leq \omega(g). \quad (2)$$

The conditional probability distributions (boxes) $\{P(a, b|x, y)\}$ are taken from sets C, Q, NS corresponding to the set of classical (LHV) boxes, quantum boxes and general no-signaling boxes, respectively. The corresponding values of the game are labeled $\omega_c(g), \omega_q(g)$ and $\omega_{ns}(g)$ respectively. We will in particular be interested in the quantum value of the game, i.e., the value obtained from those conditional probability distributions for which there exists a state $\rho \in \mathbb{B}(\mathcal{H}^d)$ and sets of measurement operators $\{M_a^x\}, \{M_b^y\}$ such that

$$P(a, b|x, y) = \text{Tr} \left(\rho M_a^x \otimes M_b^y \right). \quad (3)$$

The idea in [1] is to rewrite the general game expression in Eq.(2) as

$$\sum_{\substack{x \in \mathcal{X} \\ a \in \mathcal{A}}} \pi_A(x) P(a|x) \sum_{\substack{y \in \mathcal{Y} \\ b \in \mathcal{B}}} \pi_B(y|x) V(a, b|x, y) P(b|y, x, a).$$
$$(4)$$

One may also restrict to the scenario of the so-called *free* games for which $\pi_B(y|x) = \pi_B(y)$. Now, observe that the expression

$$\sum_{\substack{y \in \mathcal{Y} \\ b \in \mathcal{B}}} \pi_B(y) V(a, b|x, y) P(b(y)|y, x, a)_{\hat{\sigma}_{a|x}^{B}} \leq \xi_B^{(x,a)}, \quad (5)$$

has the form of a fine-grained uncertainty relation on Bob's system for each (x, a) with $\xi_B^{(x,a)}$ denoting the maximum over all possible states $\hat{\sigma}_{a|x}^{B}$ of Bob's system. For some uncertainty expressions, the optimal value

$\xi_B^{(x,a)}$ equals unity, and these we refer to as the trivial uncertainty relations, i.e., while the probabilities are bounded below unity for some states, there exists states for which the outcomes can be fixed with certainty. The uncertainty relations are illustrated in Fig. (1) and the idea of steering to the maximally certain states is illustrated in Fig. (2).
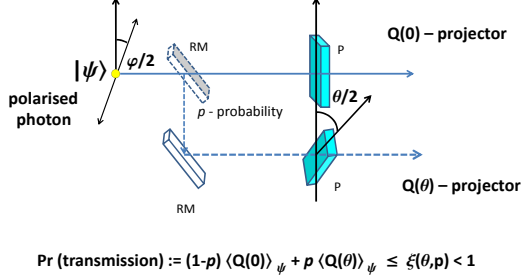

FIG. 1: The simplest uncertainty principle illustrated by randomly oriented polarisers. A reflecting mirror RM is randomly inserted with probability p on the way of a photon with polarisation described by the angle φ. One polariser P1 measures the observable $Q(0)$, and a second polariser P2 rotated by the angle $\frac{\theta}{2}$ ($0 < \theta < \pi$) with respect to the first measures $Q(\theta)$. The upper bound $\xi(\theta, p)$ on the probability that the photon is transmitted from left to right is smaller than one unless p takes the extremal value zero or one. Consequently there is no chance that the photon would be transmitted with probability one. This means that there is no quantum state co-measurable (in the sense of corresponding to the same unit eigenvalue) for the two observables, i.e., the two projectors $Q(0)$, $Q(\theta)$ corresponding to the two polarisers.

Let $\{\tilde{\sigma}_{a|x}^B\}$ denote the set of states that achieve the maximum value of the uncertainty expressions for each (x, a) for given optimal measurement operators M_b^y of Bob. The question then arises whether Alice is able to *steer* Bob's system to these maximally certain states and thus achieve the bound set by the uncertainty principle. We therefore consider the effect of steering. For any bipartite no-signaling box shared by Alice and Bob, any measurement on Alice's side creates a specific set of single-party boxes on Bob's side $\{p(b|y)_{x,a}\} = \{p(b|y,x,a)\}$. We say that with this particular input x and output a, Alice has steered the state of Bob's system to the set of boxes $\{p(b|y)_{x,a}\}$ with probability $p(a|x)$.

We see therefore in Eq.(4) the separation of the game expression into two components, one where Bob's (optimal) measurements define a set of uncertainty relations one for each (x, a) and a second component wherein Alice tries to steer Bob's system to the maximally certain
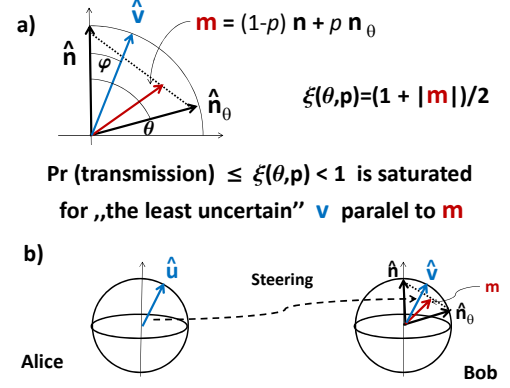

FIG. 2: a) The Bloch sphere representation of the measurement situation from the previous figure. The wavefunction $|\psi\rangle$ of the polarised photon is represented here by the vector $\hat{v}$, while the projectors $Q(0)$, $Q(\theta)$ correspond to the unit vectors $\hat{n}$, $\hat{n}_\theta$. The bound on the probability of transmission $\xi(\theta, p)$ represents a non-trivial uncertainty relation since it is strictly smaller than 1. This is a consequence of the fact that $\xi(\theta, p)$ is defined by the specific result of the convex combination in the uncertainty relation, namely the vector m, which is manifestly shorter than 1. The uncertainty relation defined by the probability of transmission is saturated by the wavefunction $|\psi\rangle$ with the Bloch vector $\hat{v}$ parallel to m. b) The situation when Alice tries to steer to the least uncertain state. She manages to achieve that iff $\hat{v} \| m$.

states corresponding to these relations. The strength of non-locality in any theory is thus seen as a trade-off between the strength of the uncertainty relations in the theory and the amount of steering allowed in the theory.

Let us remark that the above analysis may also be performed by considering a set of uncertainty relations for Alice and considering whether Bob is able to steer Alice's system to the maximally certain states for those. In other words, we write

$$\sum_{\substack{y \in \mathcal{Y} \\ b \in \mathcal{B}}} \pi_B(y) P(b|y) \sum_{\substack{x \in \mathcal{X} \\ a \in \mathcal{A}}} \pi_A(x|y) V(a, b|x, y) P(a|x, y, b),$$

(6)

and consider the set of uncertainty relations of Alice's system corresponding to each input-output pair (y, b) of Bob

$$\sum_{\substack{x \in \mathcal{X} \\ a \in \mathcal{A}}} \pi_A(x|y) V(a, b|x, y) P(a(x)|x, (y, b))_{\hat{\sigma}_{b|y}^A} \leq \xi_A^{(y,b)}. \quad (7)$$

In [1], it was shown that for all non-local games for which the optimal measurements were known, the strength of non-locality is purely determined by the uncertainty relation with steering not constraining the value in any way. In other words, the optimal measurements and the state share the property that in all

known instances, Alice (respectively Bob) is able to steer Bob's (Alice's) system to the most certain states corresponding to the set of uncertainty relations of his (her) system for each input-output pair (x, a) (respectively (y, b)). The games with the known optimal measurements referred to above are principally the two-party XOR games. These are non-local games played by two parties with an arbitrary number of inputs and binary outputs, where the winning constraint of the game only depends on the XOR of the parties' outputs. Building on a breakthrough theorem by Tsirelson [8], it was shown in [9] that the quantum value of two-party XOR games can be calculated precisely by means of a semidefinite program, and the Tsirelson theorem allows to recover the optimal state and measurement operators for any such game. In effect, apart from the pseudo-telepathy games [17] and a few other isolated instances of non-local games, these are the games where the optimal measurements are known and are the Bell inequalities where the relation between the uncertainty principle and non-locality was established in [1]. Note that the difficulty in establishing the relationship for general non-local games is due to the fact that the problem of finding the quantum value of general non-local games is NP-hard [10]; one usually uses a hierarchy of semidefinite programs formulated in [12] which converge in the limit to the true quantum value.

Relation with the Schrodinger-Hughston-Jozsa-Wootters theorem. At this point, it is worth remarking upon a curious feature of the rewriting in Eq. (4) with regard to steerability of quantum systems. Consider a set of measurement operators M_a^x on Alice's side, i.e., positive operators $M_a^x \geq 0$ satisfying $\sum_a M_a^x = \mathbf{1}$. Such a collection represents a positive-operator valued measure (POVM) for each x. For any fixed bipartite quantum state $\hat{\sigma}_{AB}$, every measurement on Alice's side gives rise to a so-called *assemblage* $\{\sigma_{a|x}^B\}_{a,x} = \{p(a|x), \hat{\sigma}_{a|x}^B\}_{a,x}$. Here

$$\sigma_{a|x}^B = \text{tr}_A \left[(M_a^x \otimes \mathbf{1}) \hat{\sigma}_{AB} \right], \qquad (8)$$

are the conditional (unnormalised) states of Bob's system prepared by Alice's measurement. We have that

$$\sum_a p(a|x) \hat{\sigma}_{a|x}^B = \sum_a p(a|x') \hat{\sigma}_{a|x'}^B = \hat{\sigma}_B = \text{tr}_A \hat{\sigma}_{AB}, \quad (9)$$

in order to obey the no-signaling principle, i.e., without the knowledge of Alice's outcome a, Bob's state is independent of the measurement choice x. The well-known Hughston-Jozsa-Wootters (HJW) theorem [13] (following from Schrödinger [11]) shows that every assemblage $\{p(a|x), \hat{\sigma}_{a|x}^B\}_{a,x}$ satisfying Eq. (9) has a quantum realization as in Eq.(8) for some quantum state $\hat{\sigma}_{AB}$ and for some set of measurement operators M_a^x. Now, coming back to the re-writing of the game expression in Eq.(4), the set of states $\{\tilde{\sigma}_{a|x}^B\}$ achieving the maximum value of the uncertainty relations in Eq.(5) together with

the optimal probabilities $\{p(a|x)\}$ forms an assemblage. One might then wonder whether the result of [1] is just a direct consequence of the HJW theorem, since Alice might steer to the assemblage corresponding to the maximally certain states. That this is *not* the case is due crucially to the fact that it is by no means guaranteed that the maximally certain states (which are obtained from a procedure of maximization of the uncertainty expressions over all quantum states of Bob's system) together with the optimal local probabilities $p(a|x)$ obey the no-signaling condition in Eq. (9). Thus, the UP-QGV correspondence found in [1] is a non-trivial property of the optimal states and measurements for some non-local games that does not automatically follow from the HJW theorem. It was posed as an open question in [1] whether the correspondence holds for all non-local games. In the following section we show that this is *not* the case by exhibiting explicit counter-examples.

Non-local games that are counter-examples to the quantum value - uncertainty principle relationship. Let us now exhibit examples of non-local games for which the above relationship between the fine-grained uncertainty principle and the quantum value of the game (UP-QGV correspondence) does not hold. In other words, we exhibit examples of games where the optimal quantum state and measurements are such that Alice is unable to steer Bob's system to the maximally certain state for each (x, a). Before we proceed to the counter-example, let us mention a further issue. It is in principle possible that the optimal quantum value of a non-local game can be achieved with different sets of states and measurement operators (even going beyond a trivial unitary equivalence), therefore one must be careful to check if the relation could hold for *at least one* set of optimal state and measurement operators for the game. Therefore, to present explicit counter-examples to the UP-QGV relationship, it is necessary to prove that the relationship does not hold for *all* optimal states and measurements for the game. We achieve this requirement by proving a self-testing property of our counter-example games, in effect there is a unique optimal state and measurements that achieves the maximal value of the game.

Counter-example Game 1. An example of a game that does not obey the relation between the uncertainty principle and the quantum value is in the scenario of two parties, each performing one of two measurements and obtaining one of two outputs; we label this Bell scenario $B(2, 2, 2)$ following the convention $B(n, m, k)$ where n denotes the number of parties, m denotes the number of inputs and k denotes the number of outputs. Alice thus receives inputs $x \in \mathcal{X} = \{0, 1\}$ and outputs $a \in \mathcal{A} = \{0, 1\}$. Similarly, Bob receives inputs $y \in \mathcal{Y} = \{0, 1\}$ and outputs $b \in \mathcal{B} = \{0, 1\}$. The winning constraint of the game which we denote $g^{(1)}$ is given by the following

predicate $V_{g^{(1)}}(a,b|x,y)$ written out in Table I

TABLE I: A game $g^{(1)}$ that violates the uncertainty principle-quantum game value correspondence

	B	y= 0		y = 1	
A		$b=0$ $\quad b=1$		$b=0$ $\quad b=1$	
	$a=0$	1	$\cdot$	$\cdot$	$\cdot$
x = 0	$a=1$	$\cdot$	$\cdot$	1	$\cdot$
	$a=0$	$\cdot$	1	$\cdot$	1
x = 1	$a=1$	1	$\cdot$	$\cdot$	$\cdot$

(10)

$$V_{g^{(1)}}(a,b|x,y) = \begin{cases} 1: & (x,y)=(0,0) \wedge (a,b)=(0,0) \\ 1: & (x,y)=(1,0) \wedge a \oplus b = 1 \\ 1: & (x,y)=(0,1) \wedge (a,b)=(1,0) \\ 1: & (x,y)=(1,1) \wedge (a,b)=(0,1) \\ 0: & \text{otherwise} \end{cases}$$

The Bell inequality for the game $g^{(1)}$ is thus explicitly given by

$$P(0,0|0,0) + P(1,0|0,1) + P(0,1|1,0)$$
$$+P(1,0|1,0) + P(0,1|1,1) \leq 2, \text{ (11)}$$

where we have assumed that each party chooses their inputs uniformly, i.e., $\pi_A(x) = \pi_B(y) = \frac{1}{2}$ for $x,y \in \{0,1\}$ so that $\pi_{AB}(x,y) = \frac{1}{4}$ and the classical bound is $4\omega_c(g^{(1)})$ with $\omega_c(g^{(1)}) = \frac{1}{2}$.

Proposition 1. *The quantum state and measurements that achieve the optimal value $\omega_q(g^{(1)})$ of the game $g^{(1)}$ do not obey the correspondence between the fine-grained uncertainty relations and the quantum value, i.e., for these optimal quantum strategies Alice is unable to steer Bob's system to the maximally certain states and vice versa.*

Examining the uncertainty relations for Alice steering Bob for each of her input-output pairs (x,a) for the game $g^{(1)}$ (given in the Supplemental Material), we see that three of the uncertainty relations are trivial, simply stating that the maximum value of the corresponding probability $P(b|y)$ is less than or equal to unity. The non-trivial uncertainty relation is the one for $(x,a) = (1,0)$, where the upper bound is strictly smaller than 1 for all states of Bob's system. The analysis in the proof of Proposition 1 given in the Supplemental Material shows that it is the trivial uncertainty relations for $(x,a) \in \{(0,0),(0,1),(1,1)\}$ that fail to be saturated, i.e., for these input-output pairs Alice is unable to steer Bob's system to the maximally certain state which is the

projector from Bob's optimal measurements associated to the corresponding (b,y). Analogous conclusions are reached when considering the steering of Alice's system by Bob by his measurements. While this shows that the UP-QGV correspondence does not hold in general, one might wonder whether there exist non-local games where the optimal measurements fail to saturate non-trivial uncertainty relations (with bounds strictly smaller than unity). We investigate this question and exhibit an example of a non-local game where even the non-trivial uncertainty relations fail to be saturated in the next section.

Counter-example Game 2. To remedy the deficiency in game $g^{(1)}$, we introduce the following game $g^{(2)}$ also in the Bell scenario $B(2,2,2)$ given by the following predicate $V_{g^{(2)}}(a,b|x,y)$

TABLE II: A game $g^{(2)}$ that violates the uncertainty principle-quantum game value correspondence

	B	y= 0		y = 1	
A		$b=0$ $\quad b=1$		$b=0$ $\quad b=1$	
	$a=0$	1	$\cdot$	$\cdot$	1
x = 0	$a=1$	$\cdot$	1	1	$\cdot$
	$a=0$	$\cdot$	1	$\cdot$	1
x = 1	$a=1$	1	$\cdot$	$\cdot$	$\cdot$

(12)

$$V_{g^{(2)}}(a,b|x,y) = \begin{cases} 1: & (x,y)=(0,0) \wedge a \oplus b = 0 \\ 1: & (x,y)=(1,0) \wedge a \oplus b = 1 \\ 1: & (x,y)=(0,1) \wedge a \oplus b = 1 \\ 1: & (x,y)=(1,1) \wedge (a,b)=(0,1) \\ 0: & \text{otherwise} \end{cases}$$

The Bell inequality for the game $g^{(2)}$ is thus given by

$$P(0,0|0,0) + P(1,1|0,0) + P(0,1|0,1) + P(1,0|0,1)$$
$$+P(0,1|1,0) + P(1,0|1,0) + P(0,1|1,1) \leq 3,$$

(13)

where we have assumed that each party chooses their inputs uniformly, i.e., $\pi_A(x) = \pi_B(y) = \frac{1}{2}$ for $x,y \in \{0,1\}$ so that $\pi_{AB}(x,y) = \frac{1}{4}$ and the classical bound is $4\omega_c(g^{(2)})$ with $\omega_c(g^{(2)}) = \frac{3}{4}$.

Proposition 2. *The quantum state and measurements that achieve the optimal value $\omega_q(g^{(2)})$ of the game $g^{(2)}$ do not obey the correspondence between the fine-grained uncertainty relations and the quantum value, i.e., for these optimal quantum strategies Alice is unable to steer Bob's system to the maximally certain states and vice versa.*

The uncertainty relations for each input-output pair (x,a) of Alice for game $g^{(2)}$ are given as

$$(x,a) = (0,0) \rightarrow$$
$$P(b=0|y=0) + P(b=1|y=1) \leq 2\xi_B^{(0,0)}$$
$$(x,a) = (0,1) \rightarrow$$
$$P(b=1|y=0) + P(b=0|y=1) \leq 2\xi_B^{(0,1)}$$
$$(x,a) = (1,0) \rightarrow$$
$$P(b=1|y=0) + P(b=1|y=1) \leq 2\xi_B^{(1,0)}$$
$$(x,a) = (1,1) \rightarrow P(b=0|y=0) \leq 1,$$

where we have used $\pi_B(y=0) = \pi_B(y=1) = \frac{1}{2}$, and the uncertainty bounds are $\xi_B^{(0,0)} \approx 0.881462$, $\xi_B^{(0,1)} \approx 0.881462$ and $\xi_B^{(1,0)} \approx 0.823244$. The optimal state and measurements achieving $\omega_q(g^{(2)})$ are given in the Supplemental Material, where it is shown explicitly that while for $(x,a) = (1,0)$ Alice steers Bob's system to the maximally certain state, for $(x,a) = (0,0)$ and $(x,a) = (0,1)$ Alice is *unable* to steer Bob's system to the maximally certain states of the corresponding (non-trivial) uncertainty relations. Further, the trivial uncertainty relation for $(x,a) = (1,1)$ also fails to be saturated. The value $\omega_q(g^{(2)})$ achievable in the quantum theory is therefore strictly lower than what is allowed by the uncertainty principle, and therefore the game $g^{(2)}$ proves explicitly that the uncertainty relation does *not* determine the non-locality of quantum theory.

Having established definitive counter-examples to the UP-QGV correspondence, let us explain why this correspondence breaks down for these specific games, and when we might expect the relationship to hold. To do so, we examine the assemblage $\{p(a|x), \tilde{\sigma}_{a|x}\}$ of maximally certain states for these games. For both games $g^{(1)}$ and $g^{(2)}$ we show that the corresponding assemblage of maximally certain states does not obey the no-signaling relation Eq.(9), so the HJW theorem does not guarantee the existence of a shared entangled state and measurements on Alice that would prepare the corresponding maximally certain states on Bob's system. Formally, we see that the UP-QGV correspondence needs to be supplemented by the following no-signaling constraint on the maximally certain states.

Observation 2. *The uncertainty principle precisely determines the non-locality of quantum theory whenever the maximally certain states $\hat{\sigma}_{a|x}^B$ of one party's measurements together with the optimal local probabilities $\{p(a|x)\}$ of the other party, forms a no-signaling assemblage, i.e., when $\{p(a|x), \hat{\sigma}_{a|x}^B\}$ obeys Eq.(9).*

The games $g^{(1)}$ and $g^{(2)}$ show that this condition is not always obeyed by the maximally certain states. An interesting open question is whether the conditions in the above Observation 2 are met for all *unique games*.

Are maximally entangled states necessary to saturate the uncertainty relations?

A notable feature of both the examples that do not obey the UP-QGV correspondence considered in the previous section, is that the optimal quantum strategy requires measurements on *non-maximally entangled* states. The XOR games considered in [1] have the property due to the Tsirelson theorem [8] that the maximum quantum violation is attained by maximally entangled states. One might intuitively expect maximally entangled states to allow for better steerability, in particular the local subsystems being totally mixed, by the HJW theorem Alice is able to steer Bob's system to *any* pure state by projecting her system onto the conjugate state. It therefore raises the question whether the UP-QGV correspondence holds only when the QGV is achieved with maximally entangled states.

The CGLMP Bell inequality (originally introduced in [14] as a 2 input, 3 output inequality) is well-known for being maximally violated by a non-maximally entangled state of two qutrits [15]. Recently in [16], this Bell inequality was shown to be *self-testing*, i.e., up to local isometries the maximal violation is achieved by a unique quantum strategy. Therefore, we investigate its UP-OGV correspondence here; the explicit expression for the CGLMP inequality and the corresponding uncertainty relations are given in the Supplemental Material. We find somewhat surprisingly, that the correspondence holds even for the non-maximally entangled state, both Alice and Bob are able to steer the other party's system to the maximally certain states for each input-output pair, despite sharing a non-maximally entangled state. An interesting open question is therefore to identify necessary and sufficient conditions for the maximally certain states to form a no-signaling assemblage as in Observation 2, for the UP-QGV correspondence to hold.

CONCLUSION AND OPEN QUESTIONS

In this paper, we have shown that the intruiging correspondence between the uncertainty principle and the quantum game value discovered in [1] does not hold in general for all non-local games. To show this, we have exhibited explicit counter-examples with fully analytical solutions of non-local games in the simplest Bell scenario, where Alice is unable to steer Bob's system to the maximally certain states corresponding to his uncertainty relations, and vice versa. We have put forth an intuitive argument for when one might expect the correspondence to hold by considering the no-signaling constraints on the maximally certain assemblages within the considerations of the Hughston-Jozsa-Wootters theorem. Finally, by examining the CGLMP Bell inequality, we have shown that despite the intuitive ease of steering with maximally entangled states, Bell inequalities

with optimal strategies achieved by non-maximally entangled states are also able to saturate the constraints imposed by the uncertainty principle.

Many interesting questions remain. Firstly, we note that the re-formulation of Bell expressions in terms of a combination of uncertainty relations and steering would appear to work best for unique games such as XOR games, where for each input-output pair of Alice and each input of Bob, the winning constraint of the game stipulates a unique output for Bob, and vice-versa. It is therefore of interest to identify counter-examples to the UP-QGV correspondence among unique games, or alternatively to identify the reasons why these games would obey this stringent relationship. Secondly, let us note that the CHSH inequality is the only facet-defining inequality in the Bell scenario $B(2,2,2)$ and the non-local games we consider are lower-dimensional faces of the classical polytope. It is important to note that while facets are the right objects of study to characterize the classical polytope, the boundary of the quantum set is a much more complicated object, and one has to consider violations of different dimensional faces in order to pick out points on the boundary. Nevertheless, it is of interest to find whether the correspondence holds for non-local games that are tight Bell inequalities (facets of the classical polytope). Thirdly, while the uncertainty relations always provide a bound on the quantum value, it is now an open question to characterize the class of games for which this bound is saturated, and investigate the quality of the bound for general Bell inequalities. Most interesting would be to find those inequalities for which the gap is extremal and identify the reasons for the same. Fourthly, one might also reverse the question and ask whether for any given entangled state, one can construct a non-local game that is uncertainty-efficient with respect to the state, i.e., for which there exist optimal measurements for this state such that the corresponding uncertainty relations are saturated. Finally, we note that for general Bell inequalities beyond non-local games, there exist different formulations of the same inequality involving different rescaling by no-signaling and normalization constraints. In this context, one might introduce quantifiers (measures) for Bell inequalities from the perspective of deviation of the assemblage of maximally certain states from no-signaling; these measures could be the tools that single out the form of general Bell inequalities that is most physically relevant as well as relevant from the perspective of information-processing protocols.

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Supplemental Material. Here, we present the formal proofs of the lemmas stated in the main text. Note that all our solutions are fully analytical, even though we use numerical values at some places to avoid cumbersome notation.

Proposition 1. The quantum state and measurements that achieve the optimal value $\omega_q(g^{(1)})$ of the game $g^{(1)}$ do not obey the correspondence between the fine-grained uncertainty relations and the quantum value, i.e., for these optimal quantum strategies Alice is unable to steer Bob's system to the maximally certain states and vice versa.

Proof. Corresponding to the Bell scenario $B(2,2,2)$ one can find the quantum set $Q(2,2,2)$ which is the set of boxes $P(a,b|x,y)$ obtained by performing measurements M_a^x, M_b^y on quantum states $\rho \in \mathbb{B}(\mathcal{H}^d)$ for arbitrary d dimensional quantum systems. In order to find the optimal value of this Bell inequality, we use the

following lemma proved by Masanes in [18].

Lemma 3 ([18]). *In the Bell scenario $B(n,2,2)$ for any number of parties n, all extreme boxes $P(a,b|x,y)$ of the quantum set $Q(n,2,2)$ can be realized by measuring n qubit pure states with projective observables.*

Since the optimal quantum value of any Bell expression in $B(2,2,2)$ is achieved at an extreme point of $Q(2,2,2)$, the Lemma 3 allows us to find the exact $\omega_q(g^{(1)})$ by optimizing over the few parameters defining projective measurements on general pure two qubit states. Let $(|\psi\rangle, \{M^x\}, \{M^y\})$ be a qubit strategy for the game $g^{(1)}$ where the projective measurement operators have the form

$$M^x = \sum_{a=0}^1 (-1)^a \Pi_a^x = \begin{pmatrix} 0 & e^{i\alpha_x} \\ e^{-i\alpha_x} & 0 \end{pmatrix}$$

$$M^y = \sum_{a=0}^1 (-1)^a \Pi_b^y = \begin{pmatrix} 0 & e^{i\beta_y} \\ e^{-i\beta_y} & 0 \end{pmatrix}$$

over the computational basis $\{|0\rangle, |1\rangle\}$, where without loss of generality $\alpha_0 = \beta_0 = 0$ and $\alpha_1, \beta_1 \in [-\pi, \pi]$. Here Π_a^x and Π_b^y are the projectors associated to the values $0,1$ obtained by diagonalizing M^x, M^y respectively.

Clearly, any qubit strategy is equivalent under local unitary transformations to such a strategy, so it suffices to compute the optimal value achieved by this strategy. We can write the Bell operator in terms of the Π_a^x and Π_b^y as

$$\mathbf{B}(g^{(1)}) = \sum_{x,y\in\{0,1\}} \pi_{AB}(x,y) \sum_{a,b\in\{0,1\}} V_{g^{(1)}}(a,b|x,y)\Pi_a^x \otimes \Pi_b^y,$$
(14)

with $\pi_{AB}(x,y) = \pi_A(x)\pi_B(y)$ for $\pi_A(x) = \pi_B(y) = 1/2$ for $x,y \in \{0,1\}$. For any set of measurements Π_a^x, Π_b^y, the optimal state $|\psi\rangle$ is the eigenvector associated to the maximum eigenvalue of the Bell operator $\tilde{\mathbf{B}}(g^{(1)}) = 4\mathbf{B}(g^{(1)})$ (where for simplicity we have explicitly absorbed the $\frac{1}{4}$ factor from the uniform input distribution $\pi_{AB}(x,y)$ into $\tilde{\mathbf{B}}$). Our task is thus to find the optimal parameters α_1, β_1 that give the maximum eigenvalue $\lambda_{max}(\tilde{\mathbf{B}}(g^{(1)}))$. Let us consider the eigenequation

$$\det[\tilde{\mathbf{B}}(g^{(1)}) - \lambda\mathbf{1}] = 0.$$
(15)

Explicitly, the eigen-equation simplifies to the following cubic equation in λ

$$2\lambda[-19 + \lambda(33 + 4\lambda(-5+\lambda))] + 2(-2+\lambda)(-1+\lambda)\left[\cos(\alpha_1) - 2\cos^2(\frac{\alpha_1}{2})\cos(\beta_1)\right] + 4 - \sin^2(\alpha_1)\sin^2(\beta_1) = 0.$$
(16)

To find the optimum quantum value $\frac{1}{4}\lambda_{max}$, we employ the Karush-Kuhn-Tucker (KKT) conditions, i.e., we investigate the expression for λ in terms of α_1 and β_1 in four sectors.

Case I: $\alpha_1 = \beta_1 = \pi$. In this case, the eigen-equation in 16 directly solves to a maximum value of $\lambda^{(I)} = 2$ corresponding to the (classical) game value of $\frac{1}{2}$.

Case II: $\beta_1 = \pi$, optimize over α_1. In this case the eigen-equation simplifies to

$$(2 - 3\lambda + \lambda^2)(1 - 4\lambda + 2\lambda^2 + \cos(\alpha_1)) = 0, \quad (17)$$

which has the four solutions $\lambda = 1, 2, (1 \pm \sin(\frac{\alpha_1}{2}))$, so that once again we obtain the maximum value of λ in this sector to be 2, corresponding to a game value of $\frac{1}{2}$.

Case III: $\alpha_1 = \pi$, optimize over β_1. In this case, the eigen-equation simplifies to

$$(\lambda - 2)^2(\lambda - 1)\lambda = 0, \quad (18)$$

giving the maximum value of λ in this sector to be 2.

Case IV: optimize over α_1, β_1. Here we have $\frac{\partial\lambda}{\partial\alpha_1} = 0$ and $\frac{\partial\lambda}{\partial\beta_1} = 0$. We implicitly differentiate the eigen-equation (16) with respect to β_1 and set $\frac{\partial\lambda}{\partial\beta_1} = 0$ to get

that either $\beta_1 = 0$ giving $\lambda = 2$ or that

$$(\lambda - 1)(\lambda - 2) = 2\sin^2\left(\frac{\alpha_1}{2}\right)\cos(\beta_1). \quad (19)$$

We now perform an implicit derivative of the eigen-equation (16) with respect to α_1 and set $\frac{\partial\lambda}{\partial\beta_1} = 0$. With the use of Eq. (19) this gives that either $\alpha_1 = 0$, i.e., $\lambda = 2$ or that the optimal values of α_1 and β_1 are related by

$$\beta_1 = \alpha_1 \pm \pi. \quad (20)$$

We now have reduced the optimization of λ to one free variable α_1. Substituting Eq. (20) into the eigen-equation (16), performing an implicit derivative with respect to α_1 and using $\frac{\partial\lambda}{\partial\alpha_1} = 0$ gives λ in terms of the α_1 as

$$\lambda = \frac{1}{2}\left(3 + 2\cos(\alpha_1) + \sqrt{2 - \cos 2\alpha_1}\right). \quad (21)$$

Substituting the above into the eigen-equation (16) and solving gives the optimal value of α_1 as

$$\alpha_1 = 2\arctan\left(\sqrt{\frac{1}{6}(5 + \sqrt{13})}\right). \quad (22)$$

The maximum eigenvalue of the Bell operator $\mathbf{B}(g^{(1)})$ is then calculated to be

$$\omega_q(g^{(1)}) = \frac{1}{4}\lambda_{max}(\tilde{\mathbf{B}}(g^{(1)})) = \frac{1}{36}(16 + \sqrt{13}) \approx 0.545. \tag{23}$$

In other words, up to local unitary transformations the quantum value of the game $g^{(1)}$ is achieved by the qubit strategy given by the measurements

$$M^x = \begin{pmatrix} 0 & e^{i\alpha_x} \\ e^{-i\alpha_x} & 0 \end{pmatrix}, M^y = \begin{pmatrix} 0 & e^{i\beta_y} \\ e^{-i\beta_y} & 0 \end{pmatrix} \tag{24}$$

with $\alpha_0 = 0, \alpha_1 = 2\arctan\left(\sqrt{\frac{1}{6}(5+\sqrt{13})}\right)$ and $\beta_0 = 0, \beta_1 = -\pi + 2\arctan\left(\sqrt{\frac{1}{6}(5+\sqrt{13})}\right)$. The corresponding optimal qubit state $|\psi\rangle$ is given by the maximal eigenvector of $\mathbf{B}(g^{(1)})$ with these measurements. Let us consider the uncertainty relations associated with the above optimal measurements. Rewriting the game expression as in Eq.(4) we see that for each choice of inputs and outputs of Alice (x,a) the following uncertainty type relations for Bob must hold.

$$\begin{aligned}
(x,a) = (0,0) &\rightarrow P(b=0|y=0) \leq 1 \\
(x,a) = (0,1) &\rightarrow P(b=0|y=1) \leq 1 \\
(x,a) = (1,0) &\rightarrow \\
P(b=1|y=0) &+ P(b=1|y=1) \leq 2\xi_B^{(1,0)} \\
(x,a) = (1,1) &\rightarrow P(b=0|y=0) \leq 1.
\end{aligned} \tag{25}$$

Here we have used that $\pi_B(y=0) = \pi_B(y=1) = \frac{1}{2}$ and

$$\xi_B^{(1,0)} = \frac{1}{4}\left(2 - e^{-i\alpha_1}\sqrt{-e^{i\alpha_1}(-1+e^{i\alpha_1})^2}\right) \approx 0.8838 \tag{26}$$

(with the optimal α_1 given in Eq.(22)) is obtained by maximization over all states of Bob's system. Note that the probability $P(b=0|y=0)$ corresponds to the projector $\Pi_{b=0}^{y=0}$ on Bob's system and $P(b=1|y=1)$ corresponds to $\Pi_{b=1}^{y=0}$ with

$$\Pi_{b=0}^{y=0} = |+\rangle\langle+|, \qquad \Pi_{b=1}^{y=0} = |-\rangle\langle-|, \tag{27}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. The probabilities $P(b=0|y=1)$ and $P(b=1|y=1)$ correspond respectively to the projectors $\Pi_{b=0}^{y=1}$ and $\Pi_{b=1}^{y=1}$ given by

$$\Pi_{b=0}^{y=1} = |+_{\beta_1}\rangle\langle+_{\beta_1}|, \qquad \Pi_{b=0}^{y=1} = |-_{\beta_1}\rangle\langle-_{\beta_1}|, \tag{28}$$

where $|+_{\beta_1}\rangle = \frac{1}{\sqrt{2}}(e^{i\beta_1}|0\rangle + |1\rangle)$, $|-_{\beta_1}\rangle = \frac{1}{\sqrt{2}}(-e^{i\beta_1}|0\rangle + |1\rangle)$, with $\beta_1 = -\pi + \alpha_1$ for α_1 given by Eq. (22).

It remains to check if the optimal state and measurements in Eq.(24) are able to achieve the maximum value of each of the uncertainty relations in Eq.(25) above. In other words, we verify whether Alice by performing the appropriate measurement x and for the prescribed output a is able to steer Bob's system to the maximally certain state that achieves the optimal value in the right hand side of the corresponding uncertainty relation. While three of the uncertainty relations in Eq.(25) correspond trivially to probabilities being bounded by unity, the relation for $(x,a) = (1,0)$ takes the form

$$\frac{1}{2}\left(\langle\Pi_{b=1}^{y=0}\rangle_{\hat{\sigma}_B} + \langle\Pi_{b=1}^{y=1}\rangle_{\hat{\sigma}_B}\right) \leq \xi_B^{(1,0)}, \tag{29}$$

and the relation holds for all quantum states $\hat{\sigma}_B$ of Bob's system. We find that for $(x,a) \in \{(0,0),(0,1),(1,1)\}$, Alice is *unable* to steer Bob's system to the maximally certain state corresponding to the associated projector $\Pi_{b=0}^{y=0}$, $\Pi_{b=0}^{y=1}$ and $\Pi_{b=0}^{y=0}$ respectively. Therefore, the quantum value of the Bell inequality is not as large as allowed by the uncertainty relations, and the steerability of the states plays a crucial role. On the other hand, for $(x,a) = (1,0)$, Alice is able to steer Bob's system to the state

$$|\psi^{(1,0)}\rangle_B = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{-e^{i\alpha_1}(-1+e^{i\alpha_1})^2}}{(-1+e^{i\alpha_1})}|0\rangle + |1\rangle\right), \tag{30}$$

with the optimal α_1 in Eq.(22) which achieves the optimal value $\xi_B^{(1,0)}$ for the corresponding uncertainty relation in Eq. (29).

Similarly, writing the game as in Eq. (4) one arrives at the following set of uncertainty relations for Alice's system for each input-output pair (y,b) of Bob

$$\begin{aligned}
(y,b) = (0,0) &\rightarrow \\
P(a=0|x=0) &+ P(a=1|x=1) \leq 2\xi_A^{(0,0)} \\
(y,b) = (0,1) &\rightarrow P(a=0|x=1) \leq 1 \\
(y,b) = (1,0) &\rightarrow P(a=1|x=0) \leq 1 \\
(y,b) = (1,1) &\rightarrow P(a=0|x=1) \leq 1,
\end{aligned} \tag{31}$$

with

$$\xi_A^{(0,0)} = \frac{1}{4}\left(2 - e^{-i\alpha_1}\sqrt{-e^{i\alpha_1}(-1+e^{i\alpha_1})^2}\right) \approx 0.8838, \tag{32}$$

and $\pi_A(x=0) = \pi_A(x=1) = \frac{1}{2}$. Again, we find that for $(y,b) \in \{(0,1),(1,0),(1,1)\}$ which give rise to trivial uncertainty relations, Bob is *unable* to steer Alice's system to the maximally certain corresponding to the projector $\Pi_{a=0}^{x=1}, \Pi_{a=1}^{x=0}$ and $\Pi_{a=0}^{x=1}$ respectively. This results in a drop in the quantum value of the game compared to a theory in which the uncertainty relation *alone* determined the quantum value of non-local games. On the other hand, for $(y,b) = (0,0)$, we find that Bob steers Alice's system to the state

$$|\psi^{(0,0)}\rangle_A = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{e^{i\alpha_1}(1-e^{i\alpha_1})^2}}{(-1+e^{i\alpha_1})}|0\rangle + |1\rangle\right), \tag{33}$$

which achieves the optimal value $\xi_A^{(0,0)}$ of the corresponding uncertainty relation

$$\frac{1}{2}\left(\langle\Pi_{a=0}^{x=0}\rangle_{\hat{\sigma}_A} + \langle\Pi_{a=1}^{x=1}\rangle_{\hat{\sigma}_A}\right) \leq \xi_A^{(0,0)}. \qquad (34)$$

We thus see that in this game, the trivial uncertainty relations are not saturated while the fine-grained relations that do not have the right hand side bound equal to unity are saturated.

Let us now show that any optimal quantum strategy for the game $g^{(1)}$ is fixed, up to local isometries to be the one given by Eq. (24). To do this, we use the following 19th century lemma by Jordan [19]

Lemma 4 (JORDAN'S LEMMA [19]). *Any two binary observables A_1 and A_2 acting on a finite-dimensional Hilbert space $\mathbf{C}^n$ with $n \geq 1$ can be simultaneously block-diagonalized into 1×1 and 2×2 blocks.*

Applying Jordan's Lemma to Alice's observables M^x for $x = 1, 2$, we obtain that

$$M^x = \oplus_i M^x(i) = \sum_i M^x(i) \otimes |i\rangle\langle i|, \qquad (35)$$

where i labels the block index and each $M^x(i)$ is a 2×2 block (the 1×1 blocks can be enhanced by adding additional dimensions without loss of generality). This gives a basis in which the Hilbert space of Alice's observables $\mathcal{H}_A$ can be written as $\mathcal{H}_A = \mathbf{C}^2 \otimes \mathcal{H}'_A$ with $\mathcal{H}'_A$ denoting the Hilbert space with basis $\{|i\rangle\}$ and $\mathbf{C}^2$ being a qubit space. A similar structure exists for the Hilbert space of Bob's observables $\mathcal{H}_B$. The crucial part of the above structure is that a measurement of the block index i commutes with both Alice's observables M^x, so one can consider a general strategy in which Alice measures $|i\rangle\langle i|$ first, and similarly Bob measures the block index for his observables M^y. This reduces the whole problem to the case where $\mathcal{H}_A = \mathbf{C}^2$ and $\mathcal{H}_B = \mathbf{C}^2$. Within the qubit subspace, we know by Lemma 3 that projective measurements on pure two qubit states achieve the extremal points, and as we have shown any qubit strategy is equivalent, up to local unitaries to the strategy

given by the measurements in Eq.(24). Therefore, there exists up to local isometries, a unique quantum strategy to achieve the optimal value of $\omega_q(g^{(1)}) = \frac{1}{36}(16 + \sqrt{13})$ for the game $g^{(1)}$. $\qquad\square$

Proposition 2. The quantum state and measurements that achieve the optimal value $\omega_q(g^{(2)})$ of the game $g^{(2)}$ do not obey the correspondence between the fine-grained uncertainty relations and the quantum value, i.e., for these optimal quantum strategies Alice is unable to steer Bob's system to the maximally certain states and vice versa.

Proof. The proof proceeds along similar lines to the proof of Proposition 1. Since the game $g^{(2)}$ also belongs to the Bell scenario $B(2, 2, 2)$, Lemma 3 allows us to find $\omega_q(g^{(2)})$ by optimizing over projective measurements on pure two qubit states. Let $(|\psi\rangle, \{M^x\}, \{M^y\})$ be a qubit strategy for the game $g^{(2)}$ with

$$M^x = \sum_{a=0}^{1}(-1)^a\Pi_a^x = \begin{pmatrix} 0 & e^{i\alpha_x} \\ e^{-i\alpha_x} & 0 \end{pmatrix}$$

$$M^y = \sum_{a=0}^{1}(-1)^a\Pi_b^y = \begin{pmatrix} 0 & e^{i\beta_y} \\ e^{-i\beta_y} & 0 \end{pmatrix} \qquad (36)$$

over the computational basis $\{|0\rangle, |1\rangle\}$, where without loss of generality $\alpha_0 = \beta_0 = 0$ and $\alpha_1, \beta_1 \in [-\pi, \pi]$. Here Π_a^x and Π_b^y are the projectors associated to the values $0, 1$ obtained by diagonalizing M^x, M^y respectively. We can write the Bell operator in terms of the Π_a^x and Π_b^y as

$$\mathbf{B}(g^{(2)}) = \sum_{x,y\in\{0,1\}} \pi_{AB}(x,y) \sum_{a,b\in\{0,1\}} V_{g^{(2)}}(a,b|x,y)\Pi_a^x\otimes\Pi_b^y$$

$$(37)$$

with $\pi_{AB}(x,y) = \pi_A(x)\pi_b(y)$ and $\pi_A(x) = \pi_B(y) = \frac{1}{2}$ for each $x, y \in \{0, 1\}$. As for game $g^{(1)}$, our task is to find the optimal α_1, β_1 that lead to the maximum eigenvalue of the operator $\tilde{\mathbf{B}}(g^{(2)}) = 4\mathbf{B}(g^{(2)})$. The eigen-equation $\det[\tilde{\mathbf{B}}(g^{(2)}) - \lambda\mathbf{1}] = 0$ simplifies to

$$\lambda\left[-30 + \lambda(33 + 2\lambda(\lambda - 7))\right] + 9 - \sin^2(\alpha_1)\sin^2(\beta_1) + (\lambda - 3)(\lambda - 1)\left[\cos(\alpha_1) - \cos(\alpha_1)\cos(\beta_1) + \cos(\beta_1)\right] = 0. \qquad (38)$$

To find the optimum quantum value $\frac{1}{4}\lambda_{max}$, we again use the KKT conditions, i.e., we investigate the expression for λ in terms of α_1 and β_1 in four sectors.

Case I: $\alpha_1 = \beta_1 = \pi$. In this case, the eigen-equation (38) directly solves to $\lambda^{(I)} = 3$ corresponding to the

(classical) game value of $\frac{3}{4}$.

Case II: $\beta_1 = \pi$, optimize over α_1. In this case the eigen-equation simplifies to

$$(\lambda^2 - 4\lambda + 3)(1 - 3\lambda + \lambda^2 + \cos(\alpha_1)) = 0, \qquad (39)$$

which has the four solutions $\lambda = 1, 3, \frac{1}{2}\left(3 \pm \sqrt{5 - 4\cos\alpha_1}\right)$, so that once again we obtain the maximum value of λ in this sector to be 3, corresponding to a game value of $\frac{3}{4}$.

Case III: $\alpha_1 = \pi$, optimize over β_1. As in the previous case II, the eigen-equation simplifies to Eq.(39) giving the maximum value of $\lambda = 3$ or a game value of $\frac{3}{4}$.

Case IV: optimize over α_1, β_1. Here we have $\frac{\partial\lambda}{\partial\alpha_1} = 0$ and $\frac{\partial\lambda}{\partial\beta_1} = 0$. We implicitly differentiate the eigen-equation (38) with respect to α_1 and set $\frac{\partial\lambda}{\partial\alpha_1} = 0$ to get

$$(\lambda-3)(\lambda-1)\sin(\alpha_1)(-1+\cos(\alpha_1)) = \sin(2\alpha_1)\sin^2(\beta_1). \tag{40}$$

Similarly implicit differentiation of Eq. (38) with respect to β_1 and setting $\frac{\partial\lambda}{\partial\beta_1} = 0$ gives

$$(3-\lambda)(\lambda-1)(1-\cos(\alpha_1)) = \sin^2(\alpha_1)\sin(2\beta_1). \tag{41}$$

Solving Eqs. (40) and (41) yields that either $\alpha_1 = 0$ giving $\lambda = 3$ or that the following relation holds between the optimal α_1 and β_1

$$\cos(\alpha_1) = \cos(\beta_1) \implies \alpha_1 = \beta_1, \tag{42}$$

since $\alpha_1, \beta_1 \in [-\pi, \pi]$. We can now simplify the eigen-equation (38) setting $\alpha_1 = \beta_1$. Implicit differentiation of the resulting expression with respect to α_1 and again setting $\frac{\partial\lambda}{\partial\alpha_1} = 0$ gives the λ in terms of the α_1 as

$$\cos\left(\frac{\alpha_1}{2}\right)\sin^3\left(\frac{\alpha_1}{2}\right)((-2+\lambda)^2 + 2\cos(\alpha_1) + \cos(2\alpha_1)) = 0,$$
$$\Rightarrow \lambda = 2 \pm \sqrt{-2\cos(\alpha_1) - \cos(2\alpha_1)}. \tag{43}$$

Substituting Eq.(43) back into the eigen-equation (38) and solving gives the optimal α_1

$$\alpha_1 = \pm 2\arctan\sqrt{\frac{\eta^2 + 2\eta - 5}{3\eta}}, \tag{44}$$

with $\eta = \left[\frac{1}{2}(43 + 9\sqrt{29})\right]^{\frac{1}{3}}$. Thus the optimal value λ_{max} is given by Eq. (43) with α_1 in Eq.(44) as

$$\begin{aligned}\omega_q(g^{(2)}) &= \frac{\lambda_{max}(\tilde{\mathbf{B}}(g^{(2)}))}{4} \\ &= \frac{1}{108}(35 + (15740 - 972\sqrt{29})^{\frac{1}{3}} \\ &\quad + 2^{\frac{2}{3}}(3935 + 243\sqrt{29})^{\frac{1}{3}}) \\ &\approx 0.7822. \end{aligned} \tag{45}$$

We thus have the optimal quantum strategy for game $g^{(2)}$ given by the measurement operators in Eq.(36) with $\alpha_1 = \beta_1$ given by Eq.(44). The corresponding optimal quantum (qubit) state is the maximal eigenvector of $\mathbf{B}(g^{(2)})$ with these measurements. By Jordan's Lemma (Lemma 4), following a similar argument to the one in the proof of Proposition 1, we have that any optimal

quantum strategy for the game $g^{(2)}$ is fixed, up to local isometries to be the one given by Eq.(36).

Let us now examine the uncertainty relations corresponding to the game $g^{(2)}$ for each input-output pair (x, a) of Alice. In what follows, we simplify notation by explicitly specifying numerical values to avoid cumbersome analytical expressions, however note that all our results are fully analytical.

$$\begin{aligned}(x,a) &= (0,0) \rightarrow \\ P(b=0|y=0) &+ P(b=1|y=1) \le 2\xi_B^{(0,0)} \\ (x,a) &= (0,1) \rightarrow \\ P(b=1|y=0) &+ P(b=0|y=1) \le 2\xi_B^{(0,1)} \\ (x,a) &= (1,0) \rightarrow \\ P(b=1|y=0) &+ P(b=1|y=1) \le 2\xi_B^{(1,0)} \\ (x,a) &= (1,1) \rightarrow P(b=0|y=0) \le 1, \end{aligned} \tag{46}$$

where we have used $\pi_B(y=0) = \pi_B(y=1) = \frac{1}{2}$, and the uncertainty bounds are

$$\begin{aligned}\xi_B^{(0,0)} &= \xi_B^{(0,1)} = \frac{1}{4}e^{-i\alpha_1}\left(2e^{i\alpha_1} - \sqrt{-e^{i\alpha_1}(-1+e^{i\alpha_1})^2}\right) \\ &\approx 0.8815, \end{aligned} \tag{47}$$

and

$$\xi_B^{(1,0)} = \frac{1}{4}e^{-i\frac{\alpha_1}{2}}\left(1 + e^{i\frac{\alpha_1}{2}}\right)^2 \approx 0.8232, \tag{48}$$

with the optimal α_1 given in Eq. (44). In
Case I: (x,a) = (0,0). The maximally certain state for $(x, a) = (0, 0)$ is

$$|\psi^{(0,0)}\rangle_A = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{e^{i\alpha_1}(1-e^{i\alpha_1})^2}}{(1-e^{i\alpha_1})}|0\rangle + |1\rangle\right). \tag{49}$$

Projecting onto the optimal state with the optimal projector $\Pi_{a=0}^{x=0} = |+\rangle\langle+|$, we see that Alice only manages to steer Bob's system to the state

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}\left(e^{-1.146i}|0\rangle + |1\rangle\right), \tag{50}$$

which achieves value $0.8446 < \xi_B^{(0,0)}$ for the corresponding uncertainty relation. Note that here we have stated the expression numerically simply to avoid the cumbersome notation associated with the exact analytical expression.

Case II: (x,a) = (0,1). The maximally certain state for $(x, a) = (0, 1)$ is

$$|\psi^{(0,1)}\rangle_A = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{e^{i\alpha_1}(1-e^{i\alpha_1})^2}}{(-1+e^{i\alpha_1})}|0\rangle + |1\rangle\right). \tag{51}$$

Projecting onto the optimal state with the optimal projector $\Pi_{a=1}^{x=0} = |-\rangle\langle-|$, we see that Alice only manages to steer Bob's system to the state

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}\left(-e^{-0.2598i}|0\rangle + |1\rangle\right),\qquad(52)$$

which achieves value $0.8446 < \xi_B^{(0,1)}$ for the corresponding uncertainty relation.

Case III: (x,a) = (1,0). The maximally certain state for $(x,a) = (1,0)$ is

$$|\psi^{(1,0)}\rangle_A = \frac{1}{\sqrt{2}}\left(-e^{i\frac{\alpha_1}{2}}|0\rangle + |1\rangle\right).\qquad(53)$$

Projecting onto the optimal state with the projector $\Pi_{a=0}^{x=1}$, we see that Alice manages to steer Bob's system to the maximally certain state in Eq.(53) for this uncertainty relation.

Case IV: (x,a) = (1,1). The maximally certain state for $(x,a) = (1,1)$ corresponds to the projector $\Pi_{b=0}^{y=0} = |+\rangle\langle+|$ Projecting onto the optimal state with the projector $\Pi_{a=1}^{x=1}$, we see that Alice only manages to

steer Bob's system to the state which achieves value $0.968 < \xi_B^{(1,1)} = 1$ for this uncertainty relation.

We thus see that for the game $g^{(2)}$, Alice is unable to steer Bob's system to the maximally certain states even for the non-trivial uncertainty relations corresponding to $(x,a) = (0,0)$ and $(x,a) = (0,1)$. The quantum value of the game $\omega_q(g^{(2)})$ is thus lower than what could have been achieved if the non-locality of the theory were bounded by the uncertainty principle alone. The same holds true from the point of view of Bob steering Alice's system. While Bob is able to steer Alice's system to the maximally certain state for $(y,b) = (1,1)$, he is unable to do so for the non-trivial uncertainty relations corresponding to $(y,b) = (0,0)$ and $(y,b) = (0,1)$ as well as for the trivial uncertainty relation for $(y,b) = (1,0)$. Thus, this example conclusively proves that the non-locality of quantum theory is not determined by the uncertainty principle alone, and steering plays a definite role. $\qquad\square$

CGLMP inequality. The CGLMP inequality is given by the following expression

$$2[P(a = b|0,0) + P(a = b|0,1) + P(a = b|1,0) + P(a = b+2|1,1)] + P(a = b+2|0,0) + P(a = b+1|0,1)$$
$$+P(a = b+1|1,0) + P(a = b+1|1,1) \le 6,\qquad(54)$$

where the addition is done modulo 3. The optimal state is

$$|\Psi\rangle = \frac{1}{\sqrt{2+\kappa^2}}\left(|00\rangle + \kappa|11\rangle + |22\rangle\right),\qquad(55)$$

and optimal measurements for the inequality are given in [15] as $\Pi_a^x = \frac{1}{3}|\psi_a^x\rangle\langle\psi_a^x|$ for Alice and $\Pi_b^y = \frac{1}{3}|\phi_b^y\rangle\langle\phi_b^y|$ for Bob with

$$|\psi_0^x\rangle = |0\rangle + e^{i\alpha_x}|1\rangle + e^{2i\alpha_x}|2\rangle$$
$$|\psi_1^x\rangle = |0\rangle + \omega e^{i\alpha_x}|1\rangle + \omega^2 e^{2i\alpha_x}|2\rangle$$
$$|\psi_2^x\rangle = |0\rangle + \omega^2 e^{i\alpha_x}|1\rangle + \omega e^{2i\alpha_x}|2\rangle,\qquad(56)$$

and

$$|\phi_0^y\rangle = |0\rangle + e^{i\beta_y}|1\rangle + e^{2i\beta_y}|2\rangle$$
$$|\phi_1^y\rangle = |0\rangle + \omega^2 e^{i\beta_y}|1\rangle + \omega e^{2i\beta_y}|2\rangle$$
$$|\phi_2^y\rangle = |0\rangle + \omega e^{i\beta_y}|1\rangle + \omega^2 e^{2i\beta_y}|2\rangle,\qquad(57)$$

with $\kappa = \frac{\sqrt{11}-\sqrt{3}}{2}$, $\omega = e^{i\frac{2\pi}{3}}$, $\alpha_0 = 0$, $\alpha_1 = \frac{\pi}{3}$, $\beta_0 = -\frac{\pi}{6}$, $\beta_1 = \frac{\pi}{6}$ [15].

For uniformly chosen inputs $\pi_A(x) = \pi_B(y) = \frac{1}{2}$ for

$x,y \in \{0,1\}$, the uncertainty relations are given explicitly by

$$(x,a) = (0,0) \to (2P(b=0|y=0) + P(b=1|y=0))$$
$$+(2P(b=0|y=1) + P(b=2|y=1)) \le \xi_B^{(0,0)}$$
$$(x,a) = (0,1) \to (2P(b=1|y=0) + P(b=2|y=0))$$
$$+(2P(b=1|y=1) + P(b=0|y=1)) \le \xi_B^{(0,1)}$$
$$(x,a) = (0,2) \to (2P(b=2|y=0) + P(b=0|y=0))$$
$$+)(2P(b=2|y=1) + P(b=1|y=1)) \le \xi_B^{(0,2)}$$
$$(x,a) = (1,0) \to (2P(b=0|y=0) + P(b=2|y=0))$$
$$+(2P(b=0|y=1) + P(b=2|y=1)) \le \xi_B^{(1,0)}$$
$$(x,a) = (1,1) \to (2P(b=1|y=0) + P(b=0|y=0))$$
$$+(2P(b=1|y=1) + P(b=0|y=1)) \le \xi_B^{(1,1)}$$
$$(x,a) = (1,2) \to (2P(b=2|y=0) + P(b=1|y=0))$$
$$+(2P(b=2|y=1) + P(b=1|y=1)) \le \xi_B^{(1,2)},$$
$$(58)$$

with $\xi_B^{(0,0)} = \xi_B^{(0,1)} = \xi_B^{(0,2)} = \xi_B^{(1,0)} = \xi_B^{(1,1)} = \xi_B^{(1,2)} = \frac{15+\sqrt{33}}{9}$. For each of these relations, we find that Alice is able to steer Bob's system to the maximally certain states

by performing her projective measurements on her part of the shared optimal two qutrit state. Similarly, for the uncertainty relations on Alice's system, Bob is able to steer Alice's system to the corresponding maximally certain states.